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On the permanental polynomials of some graphs

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Let G be a simple graph with adjacency matrix $A(G)$ and $\pi(G,x)$ the permanental polynomial of G. Let $G \times H$ denotes the Cartesian product of graphs G and H. Inspired by Klein's idea to compute the permanent of some matrices (Mol. Phy. 31 (3) (1976) 811–823), in this paper in terms of some orientation of graphs we study the permanental polynomial of a type of graphs. Here are some of our main results.

1. If G is a bipartite graph containing no subgraph which is an even subdivision of $K_{2,3}$, then G has an orientation G^e such that $\pi(G,x) = \det(xI - A(G^e))$, where $A(G^e)$ denotes the skew adjacency matrix of G^e.
2. Let G be a 2−connected outerplanar bipartite graph with n vertices. Then there exists a 2−connected outerplanar bipartite graph $\overline{G}$ with $2n+2$ vertices such that $\pi(G,x)$ is a factor of $\pi(\overline{G},x)$.
3. Let T be an arbitrary tree with n vertices. Then $\pi(T \times K_2, x) = \prod_{i=1}^{n}(x^2+1+\alpha_i^2)$, where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the eigenvalues of T.

KEY WORDS: outerplanar graph, adjacency matrix, skew adjacency matrix, characteristic polynomial, permanental polynomial, Cartesian product, Pfaffian orientation, nice cycle

1. Introduction

By a simple graph $G = (V(G), E(G))$ we mean a finite undirected graph, that is, one with no loops or parallel edges, with the vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and the edge set $E(G) = \{e_1, e_2, \ldots, e_m\}$, if not specified. The adjacency matrix of a graph G, here denoted by $A(G) = (a_{ij})_{n\times n}$, is a matrix of order n whose entries $a_{ij} = 1$ if vertex v_i is adjacent to vertex v_j and $a_{ij} = 0$

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otherwise. Obviously, $A(G)$ is a symmetric matrix whose trace equals zero. The characteristic polynomial of a graph G is, by definition

$$\phi(G, x) = \det(xI - A(G)), \tag{1}$$

where I is the unit matrix of order n. The permanental polynomial of G, denoted by $\pi(G, x)$, is defined as

$$\pi(G, x) = \mathrm{per}(xI - A(G)), \tag{2}$$

where $\mathrm{per}(X)$ denotes the permanent of matrix X.

We will denote the characteristic polynomial and permanental polynomial of a graph G in the coefficient forms as follows.

$$\phi(G, x) = \det(xI - A(G)) = \sum_{k=0}^{n} a_k x^{n-k}, \tag{3}$$

$$\pi(G, x) = \mathrm{per}(xI - A(G)) = \sum_{k=0}^{n} b_k x^{n-k}. \tag{4}$$

If G is a bipartite graph, it is not difficult to see [1–5] that

$$(-1)^k a_{2k} \geqslant 0,\ \ b_{2k} \geqslant 0,\ \ a_{2k+1} = b_{2k+1} = 0 \text{ for all } k \geqslant 0, \tag{5}$$

and b_{2k} equals $\sum_H \mathrm{per}(A(H))$, where $A(H)$ is the adjacency matrix of the induced subgraph H of G with $2k$ vertices and the sum ranges over all induced subgraphs of G with $2k$ vertices. Hence if G is a bipartite graph we may write $\phi(G, x)$ and $\pi(G, x)$ as follows.

$$\phi(G, x) = \sum_{k=0}^{[n/2]} a_{2k} x^{n-2k}, \tag{6}$$

$$\pi(G, x) = \sum_{k=0}^{[n/2]} b_{2k} x^{n-2k}, \tag{7}$$

where $[n/2]$ denotes the greatest integer no more than $n/2$.

Note that the characteristic polynomial of graphs and its applications are extensively examined (see for example [6, 7, 28, 29]). However, with some exceptions [1–3,7–14], little about the permanental polynomial and its potential applications seems to have been published [8]. This may be due to the difficult to actually computing the permanent $\mathrm{per}(xI - A(G))$. Many shortcuts exist for computing determinants of matrices, whereas only a few methods exist for permanents [6,12,14–19,27].

Bearing in mind the definition of a permanent, for the coefficients of the permanental polynomial in equation (4) one has [3, 5]

$$(-1)^k b_k = \sum_S 2^{c(S)}, \tag{8}$$

where S is a Sachs subgraph of G with k vertices (a subgraph S is called a Sachs subgraph if all components of S are edges or cycles) and $c(S)$ denotes the number of cycles in S, and the sum ranges over all Sachs subgraphs of G with k vertices. Perhaps this is the earliest formula for the coefficients of the permanental polynomial of a graph G. Obviously, this is not a good method to compute the permanental polynomial of graphs. In fact since the complexity of computation of the permanent of matrices is NP-complete, so is the complexity of the computation of the permanental polynomial of graphs (see [20]). Hence it is interesting to find methods to compute the permanental polynomial of some type of graphs.

If G is a tree with n vertices, Merris et al. [5] (see also [4]) proved that if

$$\phi(G, x) = \sum_{k=0}^{[n/2]} a_{2k} x^{n-2k} \quad \text{then} \quad \pi(G, x) = \sum_{k=0}^{[n/2]} (-1)^k a_{2k} x^{n-2k}. \tag{9}$$

This result was generalized by Borowiechi [1] (see also [4]) as follows. If G is a bipartite graph containing no cycle of length $4s, s \in \{1, 2, \dots\}$ and $\phi(G, x) = \sum_{k=0}^{[n/2]} a_{2k} x^{n-2k}$ then $\pi(G, x) = \sum_{k=0}^{[n/2]} (-1)^k a_{2k} x^{n-2k}$. Note that the characteristic polynomial can be computed easily. Hence these results show that the permanental polynomial of a tree or a bipartite graph containing no cycle of length $4s$, $s \in \{1, 2, \dots\}$ can be computed easily. Recently, the relation between permanental and characteristic polynomials of some chemical graphs were considered in [4, 8–10].

In order to formulate our main results, we need to introduce some notation. Let G be a simple graph. A set M of edges in G is a matching if every vertex of G is incident with at most one edge in M; it is a perfect matching if every vertex of G is incident with exactly one edge in M. We denote by $M(G)$ the number of perfect matchings of G. If M is a perfect matching of G, an M-alternating cycle in G is a cycle whose edges are alternately in $E(G)\backslash M$ and M. Let G be a graph. We say that a cycle C of G is nice if $G - C$ contains a perfect matching, where $G - C$ denotes the induced subgraph of G obtained from G by deleting the vertices of C. Let G^e be an arbitrary orientation of G. The skew adjacency matrix of G^e, denoted by $A(G^e)$, is defined as follows:

$$A(G^e) = (b_{ij})_{n\times n}, \qquad b_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in E(G^e), \\ -1 & \text{if } (v_j, v_i) \in E(G^e), \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Hence the skew adjacency matrix $A(G^e)$ is a skew symmetric matrix, that is, $(A(G^e))^{\mathrm{T}} = -A(G^e)$, where B^{T} denotes the transpose of the matrix B.

Let G be a simple graph and let $G_0, G_1, G_2, \dots, G_k$ be graphs such that $G_0 = G$ and, for each $i > 0$, G_i can be obtained from G_{i-1} by subdividing an edge twice. Then G_k is said to be an even subdivision of G. A plane graph G is called to be outerplanar if it is planar and embeddable into the plane such that all vertices lie on the outer face. Throughout this paper, $G \times H$ denotes the Cartesian product of two graphs G and H. We denote the complete graph with n vertices by K_n and the complete bipartite graph by $K_{s,t}$.

In this paper, inspired by the idea to compute the permanent of some matrices in [12], we consider the permanental polynomial of some graphs. We will prove the following results.

(1) If G is a bipartite graph containing no subgraph which is an even subdivision of $K_{2,3}$, then G has an orientation G^e such that $\pi(G, x) = \det(xI - A(G^e))$, where $A(G^e)$ denotes the skew adjacency matrix of G^e.
(2) Let G be a 2-connected outerplanar bipartite graph with n vertices. Then there exists a 2-connected outerplanar bipartite graph $\overline{G}$ with $2n + 2$ vertices such that $\pi(G, x)$ is a factor of $\pi(\overline{G}, x)$.
(3) Let T be an arbitrary tree with n vertices. Then $\pi(T \times K_2, x) = \prod_{i=1}^{n}(x^2 + 1 + \alpha_i^2)$, where $\alpha_1, \alpha_2, \dots, \alpha_n$ are the eigenvalues of T.

2. Preliminaries

If D is an orientation of a simple graph G and C is a cycle of even length, we say that C is oddly oriented in D if C contains odd number of edges that are directed in D in the direction of each orientation of C. We say that D is a Pfaffian orientation of G if every nice cycle of even length of G is oddly oriented in D. Kasteleyn [21] introduced a remarkable method for enumerating perfect matchings which reduces the enumeration to the evaluation of the determinant of the skew adjacency matrix of the Pfaffian orientation of G as follows.

Lemma 2.1. [22] Let G^e be a Pfaffian orientation of a graph G. Then

$$M^2(G) = \det A(G^e),$$

where $A(G^e)$ is the skew adjacency matrix of G^e.

Lemma 2.2. [22] Let G be a connected plane graph, and G^e an orientation of G such that every boundary face – except possibly the infinite face – has an odd number of edges oriented clockwise. Then in every cycle of G^e the number of edges oriented clockwise is of opposite parity to the number of vertices of G^e inside the cycle. Consequently, G^e is a Pfaffian orientation of G. Furthermore, such an orientation can be constructed in polynomial time.

Lemma 2.3. [22] Let G be any simple graph with even number of vertices, and G^e an orientation of G. Then the following three properties are equivalent:

(1) G^e is a Pfaffian orientation.
(2) Every nice cycle of even length in G is oddly oriented in G^e.
(3) If G contains a perfect matching, then for some perfect matching F, every F-alternating cycle is oddly oriented in G^e.

Lemma 2.4. [23] If G is a simple graph containing no subgraph which is, after the contraction of at most one cycle of odd length, an even subdivision of $K_{2,3}$, then G has an orientation under which every cycle of even length is oriented oddly. Furthermore, such an orientation is a Pfaffian orientation of G.

The above lemma was given by Fischer and Little [23] in 2002. In fact, they gave a more general characterization of graphs that have an orientation under which every cycle of even length has a prescribed parity.

Lemma 2.5. [24] Let T be an arbitrary tree. Then every cycle of $T \times K_2$ is a nice cycle.

Let T be a tree with the vertex set $V(T) = \{v_1, v_2, \ldots, v_n\}$ and T^e an arbitrary orientation of T. Take two copies of T, denoted by T_1 with the vertex set $V(T_1) = \{v_1', v_2', \ldots, v_n'\}$ and T_2 with the vertex set $V(T_2) = \{v_1'', v_2'', \ldots, v_n''\}$, and add an edge $v_i'v_i''$ between every pair of corresponding vertices v_i' and v_i'' for $1 \leqslant i \leqslant n$, respectively. Then the resulting graph is $T \times K_2$. It is obvious that all adding edges $v_i'v_i''$ for $1 \leqslant i \leqslant n$ in $T \times K_2$ form a perfect matching. If we define the orientation T_1^e of T_1 in $T \times K_2$ to be T^e and the orientation T_2^e of T_2 in $T \times K_2$ to be the converse of T^e, that is, the orientation by reversing the orientation of each arc of T^e, and the directions of edges $v_i'v_i''$ for $1 \leqslant i \leqslant n$ in $T \times K_2$ to be from v_i' to v_i''. Then we obtain an orientation of $T \times K_2$ from the orientation T^e of T, denoted here by $(T \times K_2)^e$. Figure 1 illustrates this procedure.

In terms of Lemmas 2.3 and 2.5, Yan and Zhang [24] or Yan [25] proved the following lemma.

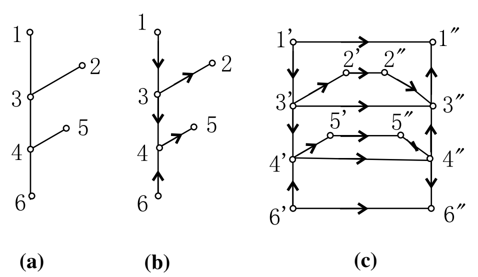

Figure 1. (a) A tree T; (b) an orientation T^e of T; (c) the orientation $(T \times K_2)^e$ of $T \times K_2$.

Lemma 2.6. [24,25] Let T be an arbitrary tree and T^e an arbitrary orientation of T. Then $(T \times K_2)^e$ defined above is a Pfaffian orientation of $T \times K_2$ under which every cycle of $(T \times K_2)^e$ is oddly oriented.

Lemma 2.7. [22] Let G be a bipartite graph and $A(G)$ the adjacency matrix of G. Then

$$\mathrm{per}(A(G)) = M^2(G), \tag{11}$$

where $M(G)$ is the number of perfect matchings of G.

3. Main results

Although the complexity of computation of permanent of the square matrices is NP-complete. Klein [12] found a method to compute the permanent of the adjacency matrix of a type of graphs – outerplanar bipartite graphs. The key to this method is to orient the outerplanar bipartite graphs such that every cycle in this orientation is oriented oddly. Then the permanent of the adjacency matrix equals the determinant of a skew adjacency matrix. The following theorem shows that this method can also be used to compute the permanental polynomial of a type of graphs.

Theorem 3.1. Let G be a bipartite graph with n vertices containing no subgraph which is an even subdivision of $K_{2,3}$. Then there exists an orientation G^e of G such that the permanental polynomial of G

$$\pi(G, x) = \det(xI - A(G^e)), \tag{12}$$

where $A(G^e)$ denotes the skew adjacency matrix of G^e.

Proof. Since G is a bipartite graph containing no subgraph which is an even subdivision of $K_{2,3}$, by Lemma 2.4, G has an orientation G^e under which every cycle is oriented oddly. Let $A(G^e)$ be the skew adjacency matrix of G^e and

$$\psi(G, x) = \det(xI - A(G^e)) = \sum_{k=0}^{n} c_k x^{n-k}. \tag{13}$$

Hence we only need to prove that $\pi(G, x) = \psi(G, x)$.

Note that by equations (5) and (7) the permanental polynomial $\pi(G, x)$ of G has the following form.

$$\pi(G, x) = \sum_{k=0}^{[\frac{n}{2}]} b_{2k} x^{n-2k}, \tag{14}$$

where $b_{2k} = \sum_H \mathrm{per}(A(H))$, and the sum ranges over all induced subgraphs H of G with $2k$ vertices and $A(H)$ is the adjacency matrix of H. Hence, by Lemma 2.7, we have

$$b_{2k} = \sum_H M^2(H), \tag{15}$$

where the sum is over all induced sugraphs of G with $2k$ vertices and $M(H)$ denotes the number of perfect matchings of H. So it is suffice for us to prove that for all $k \geqslant 0$,

$$c_{2k+1} = 0, \quad c_{2k} = \sum_H M^2(H). \tag{16}$$

Note that $(-1)^k c_k$ equals the sum of $(-1)^k \det(A(H^e))$ over all induced subdigraphs H^e of G^e with k vertices, where $A(H^e)$ is the skew adjacency matrix of sbdigraph H^e. Hence when k is odd we have $c_k = 0$ since the determinant of a skew symmetric matrix of odd order equals zero. So we assume that k is even. Let H be the underlying graph of H^e. Since every cycle of G is oriented oddly in G^e, every cycle of H is oriented oddly in H^e. Hence H^e is a Pfaffian orientation of H. Then, by Lemma 2.1, if k is even we have

$$c_k = (-1)^k \sum_H \det(A(H^e)) = \sum_H \det(A(H^e)) = \sum_H M^2(H) \quad \text{for all} \quad k \geqslant 0. \tag{17}$$

The theorem is thus proved. □

Remark 1. If a bipartite graph G has an orientation G^e such that every cycle in G is oddly oriented in G^e then this orientation works for the statement of Theorem 3.1.

Corollary 3.2. Let G be a outerplanar bipartite graph. Then G has an orientation G^e such that

$$\pi(G, x) = \det(xI - A(G^e)). \tag{18}$$

Furthermore, such an orientation can be constructed in polynomial time.

Proof. Note that G is a plane graph. Hence, by Lemma 2.2, we can construct an orientation G^e of G in polynomial time under which every boundary face – except possibly the infinite face – has and odd number of edges oriented clockwise. Furthermore, in every cycle the number of edges oriented clockwise is of opposite parity to the number of vertices of G^e inside the cycle. Since G is a outerplanar bipartite, inside every cycle in G there exists no vertex. Hence every cycle in G^e is oddly oriented. Then, by Remark 1, we have

$$\pi(G, x) = \det(xI - A(G^e)). \tag{19}$$

The corollary is proved. □

Corollary 3.3. Let G be an even subdivision of a outerplanar bipartite graph. Then G has an orientation G^e such that

$$\pi(G, x) = \det(xI - A(G^e)). \tag{20}$$

Proof. Note that since G is an even subdivision of a outerplanar bipartite graph there exist even number of vertices inside very cycle of G. Then similarly to that in Corollary 3.2 we can prove Corollary 3.3. □

Remark 2. Corollary 3.2 is a special case of Corollary 3.3.

Corollary 3.4. Let G be a bipartite graph containing no subgraph which is an even subdivision of $K_{2,3}$. Then the roots of the permanental polynomial $\pi(G, x)$ of G are pure imaginary or zero.

Proof. By Theorem 3.1, G has an orientation G^e such that $\pi(G, x) = \det(xI - A(G^e))$. Note that $A(G^e)$ is a skew adjacency matrix. Since eigenvalues of a skew adjacency matrix are pure imaginary or zero, the roots of $\pi(G, x)$ are pure imaginary or zero. The corollary thus follows.

Cash [8] said that Klein observed the fact that the roots of the permanental polynomial of all outerplanar bipartite graphs were pure imaginary or zero. Now we generalize this result as follows. □

Corollary 3.5. Let G be an even subdivision of a outerplanar bipartite graph. Then the roots of the permanental polynomial $\pi(G, x)$ are pure imaginary or zero.

Proof. Similarly to the proof of Corollary 3.4, by Corollaris 3.2 and 3.3 we can prove Corollary 3.5. □

Remark 3. Let G be a graph with n vertices such that the roots of $\phi(G, x)$ are $x_1, x_2, \ldots, x_n$. Borowiechi [1, 2] characterized all graphs the roots of whose permanental polynomial are $ix_1, ix_2, \ldots, ix_n$, where $i^2 = -1$. He proved that if the roots of $\phi(G, x)$ are $x_1, x_2, \ldots, x_n$ then the roots of $\pi(G, x)$ are $ix_1, ix_2, \ldots, ix_n$ if and only if G is a bipartite graph containing no cycle of length $4s$, $s \in \{1, 2, \ldots\}$. Furthermore, Borowiecki and Jóżwiak [2] posed the following problem: Characterize those graphs the roots of whose permanental polynomial are pure imaginary or zero. It is obvious that Corollaries 3.4 and 3.5 are a partial solution of Borowiecki and Jóżwiak's problem.

In [8] Cash identified properties and uses of the permanental polynomial of some unweighted chemical graphs. In terms of Mathematica, with exceptions he found no permanental polynomial of graphs had factors smaller than themselves (that is, factors with integer coefficients; obviously, any polynomial with roots r_i has factors $x - r_i$). On the other hand, the following theorem shows that there exist many bipartite graphs whose permanental polynomial has factor with integer coefficients smaller than themselves, that is, we will prove that the permanental polynomial of any 2-connected outerplanar bipartite graph G with n vertices can be a factor of the permanental polynomial of a 2-connected outerplanar bipartite graph with $2n + 2$ vertices.

Theorem 3.6. Let G be a 2-connected outerplanar bipartite graph with n vertices. Then there exists a 2-connected bipartite graph $\overline{G}$ with $2n+2$ vertices such that $\pi(G, x)$ is a factor of $\pi(\overline{G}, x)$.

Proof. Take two copies of G, denoted by G_1 with the vertex set $V(G_1) = \{v_1', v_2', \dots, v_n'\}$ and G_2 with the vertex set $V(G_2) = \{v_1'', v_2'', \dots, v_n''\}$. Obviously, the mapping $f : v_i' \longmapsto v_i''$ is an isomorphism between G_1 and G_2. For the sake of convenience, we assume that $v_1'v_2'$ is an edge of G_1 lying on the infinite boundary face. Let $\overline{G}$ be the graph obtained from G, which has the vertex set $V(\overline{G}) = \{v_1', v_2', \dots, v_n', v_1'', v_2'', \dots, v_n'', a, b\}$ and the edge set $E(\overline{G}) = E(G_1) \cup E(G_2) \cup \{v_1'a, v_1''b, v_2'a, v_2''b\}$. Figure 2 shows this procedure constructing graph $\overline{G}$ from G.

It is not difficult to see that since G is a 2-connected outerplanar bipartite graph with n vertices $\overline{G}$ is a 2-connected outerplanar bipartite graph with $2n+2$ vertices. Hence it is suffice to prove that $\pi(G, x)$ is a factor of $\pi(\overline{G}, x)$.

Note that G is an outerplanar bipartite graph. As that in the proof of Corollary 3.2, G has an orientation G^e such that every cycle of G is oddly oriented in G^e. Let $\overline{G}^e$ be the orientation of $\overline{G}$ which is obtained from G^e by defining the orientations of the induced subgraphs G_1 and G_2 of $\overline{G}$ to be G^e, and the directions of edges $v_1'a, v_1''a, v_2'b, v_2''b$ in $\overline{G}^e$ to be from v_1' to a, v_1'' to a, v_2' to b, v_2'' to b, respectively (see Figure 3). It is not difficult to show that $\overline{G}^e$ is such an orientation that every cycle of $\overline{G}$ is oddly oriented.

Let $A(G^e)$ be the skew adjacency matrix of G^e. Then, by a suitable labelling of vertices of $\overline{G}^e$, the skew adjacency matrix $A(\overline{G}^e)$ of $\overline{G}^e$ has the following form.

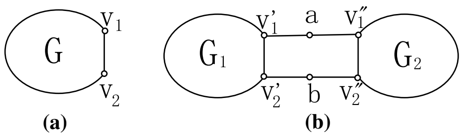

Figure 2. (a) The graph G; (b) the graph $\overline{G}$ obtained from G.

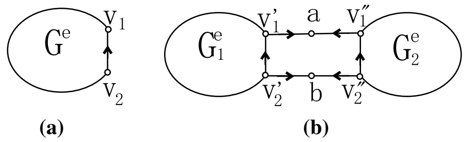

Figure 3. (a) The orientation G^e of G; (b) the orientation $\overline{G}^e$ obtained from G^e.

$$A(\overline{G}^e) = \begin{bmatrix} A(G^e) & B & 0 \\ -B^{\mathrm{T}} & 0 & -B^{\mathrm{T}} \\ 0 & B & A(G^e) \end{bmatrix}, \tag{21}$$

where B denotes the incident relation between G_1^e and $\{a, b\}$ in $\overline{\mathrm{G}}^e$, and B^T is the transpose of matrix B. Hence, by Remark 1, we have

$$\begin{aligned} \pi(\overline{G}, x) = \det(xI_{2n+2} - A(\overline{G}^e)) &= \det \begin{bmatrix} xI_n - A(G^e) & -B & 0 \\ B^{\mathrm{T}} & xI_2 & B^{\mathrm{T}} \\ 0 & -B & xI_n - A(G^e) \end{bmatrix} \\ &= \det \begin{bmatrix} xI_n - A(G^e) & -B & 0 \\ 2B^{\mathrm{T}} & xI_2 & B^{\mathrm{T}} \\ xI_n - A(G^e) & -B & xI_n - A(G^e) \end{bmatrix} \\ &= \det \begin{bmatrix} xI_n - A(G^e) & -B & 0 \\ 2B^{\mathrm{T}} & xI_2 & B^{\mathrm{T}} \\ 0 & 0 & xI_n - A(G^e) \end{bmatrix} \\ &= \det(xI_n - A(G^e)) \det \begin{bmatrix} xI_n - A(G^e) & -B \\ 2B^{\mathrm{T}} & xI_2 \end{bmatrix} \\ &= \pi(G, x) \det \begin{bmatrix} xI_n - A(G^e) & -B \\ 2B^{\mathrm{T}} & xI_2 \end{bmatrix}. \end{aligned}$$

Note that

$$\det \begin{bmatrix} xI_n - A(G^e) & -B \\ 2B^{\mathrm{T}} & xI_2 \end{bmatrix} \tag{22}$$

is a factor with integer coefficients of $\pi(\overline{G}, x)$. The theorem is thus proved.

The following lemma is useful, which was proved by Yan and Zhang [24] or Yan [25].

Lemma 3.7. [24,25] Let T be a tree with n vertices and T^e an arbitrary orientation of T. Then the eigenvalues of $A(T^e)$ are of the form $i\alpha_1, i\alpha_2, \ldots, i\alpha_n$, where $\alpha_1, \alpha_2, \ldots, \alpha_n$ are the eigenvalues of $A(T)$, and $i^2 = -1$.

Theorem 3.8. Let T be an arbitrary tree with n vertices and $T \times K_2$ the Cartesian product of graphs T and K_2 (the complete graph with two vertices). Then the permanental polynomial of $T \times K_2$

$$\pi(T \times K_2, x) = \prod_{\alpha}(x^2 + 1 + \alpha^2), \tag{23}$$

where the product ranges over all eigenvalues α of T.

Proof. Let T^e be an arbitrary orientation of T and $(T \times K_2)^e$ the orientation of $T \times K_2$ defined in Lemma 2.6 (see Figure 1). By Lemma 2.6, every cycle of $T \times K_2$ is oddly oriented in $(T \times K_2)^e$. Then, by Remark 1, we have

$$\pi(T \times K_2, x) = \det(xI_{2n} - A((T \times K_2)^e)), \tag{24}$$

where $A((T \times K_2)^e)$ denotes the skew adjacency matrix of $(T \times K_2)^e$, and I_{2n} is the unit matrix of order $2n$.

By a suitable labelling of vertices of $(T \times K_2)^e$, the skew adjacency matrix of $(T \times K_2)^e$ has the following form.

$$A((T \times K_2)^e) = \begin{bmatrix} A(T^e) & I_n \\ -I_n & -A(T^e) \end{bmatrix}, \tag{25}$$

where $A(T^e)$ is the skew adjacency matrix of T^e. Hence we have

$$\begin{aligned} \pi(T \times K_2, x) &= \det \begin{bmatrix} xI_n - A(T^e) & -I_n \\ I_n & xI_n + A(T^e) \end{bmatrix} \\ &= \det(x^2 I_n - A^2(T^e) + I_n) \\ &= \det((x^2+1)I_n - A^2(T^e)). \end{aligned}$$

Suppose that the eigenvalues of $A(T)$ are $\alpha_1, \alpha_2, \ldots, \alpha_n$. Then, by Lemma 3.7, $A(T^e)$ has eigenvalues $i\alpha_1, i\alpha_2, \ldots, i\alpha_n$, where $i^2 = -1$. Hence we have

$$\pi(T \times K_2, x) = \det((x^2+1)I_n - A^2(T^e)) = \prod_{k=1}^{n}(x^2 + 1 + \alpha_k^2). \tag{26}$$

Thus the theorem is proved. □

Remark 4. Note that Schwenk [26] ever proved that almost all trees have cospectral trees. Merris et al. [5] used this fact to show that almost all trees have copermanental trees (two graphs G and H are called to be copermanental if $\pi(G, x) = \pi(H, x)$ and G and H are not isomorphic). Obviously, by Theorem 3.8, there exist infinite many pair of 2-connected bipartite graphs which are copermanental, since if T_1 and T_2 are two cospectral trees then by Theorem 3.8 $T_1 \times K_2$ and $T_2 \times K_2$ are copermanental.

Corollary 3.9. Let T be an arbitrary tree with n vertices and $T \times K_2$ the Cartesian product of graphs T and K_2 (the complete graph with two vertices). Then the permanental polynomial of $T \times K_2$

$$\pi(T \times K_2, x) = (x^2+1)^{n-2r}[a_0(x^2+1)^r - a_1(x^2+1)^{r-1} + \cdots + (-1)^r a_r]^2, \quad (27)$$

where r is the maximum number of edges in a matching of T, and the characteristic polynomial of T is $\phi(T,x) = x^{n-2r}\sum_{j=0}^{r} a_j x^{2r-2j}$, and $a_0 = 1$.

Proof. Note that $\phi(T,x) = x^{n-2r}\sum_{j=0}^{r} a_i x^{2r-2j} = x^{n-2r}\prod_{j=1}^{r}(x^2 - \alpha_j^2)$, where the positive eigenvalues of T are $\alpha_1, \alpha_2, \ldots, \alpha_r$. Since the spectrum of T is symmetric with respect to zero, by Theorem 3.8 we have

$$\pi(T \times K_2, x) = (x^2+1)^{n-2r}\prod_{j=1}^{r}(x^2+1+\alpha_j^2)^2. \quad (28)$$

Note that

$$\phi^2(T, ix) = \{(ix)^{n-2r}\prod_{j=1}^{r}[(ix)^2 - \alpha_j^2]\}^2 = (-1)^{n-2r}(x^2)^{n-2r}\prod_{j=1}^{r}(x^2+\alpha_j^2)^2,$$

where $i^2 = -1$. Hence by equation (28) we have

$$\begin{aligned}\pi(T \times K_2, x) &= (-1)^{n-2r}\phi^2(T, i\sqrt{x^2+1}) \\ &= (x^2+1)^{n-2r}[(x^2+1)^r - a_1(x^2+1)^{r-1} + \cdots + (-1)^r a_r]^2.\end{aligned}$$

The corollary thus follows. □

Remark 5. In Corollary 3.9, if n is even or $n = 2r$, then $\pi(T \times K_2, x)$ can be denoted by the square of a polynomial with integer coefficients. This fact shows again that there exist many 2-connected bipartite graphs whose permanental polynomial has factor with integer coefficients smaller than themselves.

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